
CONSTRUCTIBLE CHARACTERS AND b -INVARIANT

by

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Let (W, S) be a finite Coxeter system and let $\varphi : S \rightarrow \mathbb{R}_{>0}$ be a *weight function* that is, a map such that $\varphi(s) = \varphi(t)$ whenever s and t are conjugate in W . Associated with this datum, G. Lusztig has defined [Lu3, §22] a notion of *constructible characters* of W : it is conjectured that a character is constructible if and only if it is the character afforded by a Kazhdan-Lusztig left cell (defined using the weight function φ). These constructible characters depend heavily on φ so we will call them the φ -*constructible characters* of W : the set of φ -constructible characters will be denoted by $\text{Cons}_{\varphi}^{\text{Lus}}(W)$. We shall also define a graph $\mathcal{G}_{W, \varphi}^{\text{Lus}}$ as follows: the vertices of $\mathcal{G}_{W, \varphi}^{\text{Lus}}$ are the irreducible characters and two irreducible characters χ and χ' are joined in this graph if there exists a φ -constructible character γ of W such that χ and χ' both occur as constituents of γ . The connected components of $\mathcal{G}_{W, \varphi}^{\text{Lus}}$ (viewed as subsets of $\text{Irr}(W)$) will be called the *Lusztig φ -families*: the set of Lusztig φ -families will be denoted by $\text{Fam}_{\varphi}^{\text{Lus}}(W)$. If $\mathcal{F} \in \text{Fam}_{\varphi}^{\text{Lus}}(W)$, we denote by $\text{Cons}_{\varphi}^{\text{Lus}}(\mathcal{F})$ the set of φ -constructible characters of W all of whose irreducible components belong to \mathcal{F} .

On the other hand, using the theory of rational Cherednik algebras at $t = 0$ and the geometry of the Calogero-Moser space associated with (W, φ) , R. Rouquier and the author (see [BoRo1] and [BoRo2]) have defined a notion of *Calogero-Moser φ -cells* of W , a notion of *Calogero-Moser φ -cellular characters* of W (whose set is denoted by $\text{Cell}_{\varphi}^{\text{CM}}(W)$) and a notion of *Calogero-Moser φ -families* (whose set is denoted by $\text{Fam}_{\varphi}^{\text{CM}}(W)$).

Conjecture (see [BoRo1], [BoRo2] and [GoMa]). *With the above notation,*

$$\text{Cons}_{\varphi}^{\text{Lus}}(W) = \text{Cell}_{\varphi}^{\text{CM}}(W) \quad \text{and} \quad \text{Fam}_{\varphi}^{\text{Lus}}(W) = \text{Fam}_{\varphi}^{\text{CM}}(W)$$

for every weight function $\varphi : S \rightarrow \mathbb{R}_{>0}$.

The statement about families in this conjecture holds for classical Weyl groups thanks to a case-by-case analysis relying on [Lu3, §22] (for the computation of Lusztig φ -families), [GoMa] (for the computation of Calogero-Moser φ -families in type A and B) and [Be2] (for the computation of the Calogero-Moser φ -families in type D). It also holds whenever $|S| = 2$ (see [Lu3, §17 and Lemma 22.2] and [Be1, §6.10]).

The statement about constructible characters is much more difficult to establish, as the computation of Calogero-Moser φ -cellular characters is at that time out of reach: it has been proved whenever the Calogero-Moser space associated with (W, S, φ) is smooth [BoRo2, Theorem 14.4.1] (it has also been checked if W is of type B_2 ...).

Our aim in this paper is to show that this conjecture is compatible with properties of the b -invariant (as defined below). With each irreducible character χ of W is associated its *fake degree* $f_\chi(\mathbf{t})$, using the invariant theory of W (see for instance [BoRo2, Definition 1.5.7]). Let us denote by b_χ the valuation of $f_\chi(\mathbf{t})$: b_χ is called the *b -invariant* of χ . For instance, $b_1 = 0$ and b_ε is the number of reflections of W (here, $\varepsilon : W \rightarrow \{1, -1\}$ denotes the sign character). Also, $b_\chi = 1$ if and only if χ is an irreducible constituent of the canonical reflection representation of W . The following result is proved in [BoRo2, Theorems 9.6.1 and 12.3.14]:

Theorem CM. *Let $\varphi : S \rightarrow \mathbb{R}_{>0}$ be a weight function. Then:*

- (a) *If $\mathcal{F} \in \text{Fam}_\varphi^{\text{CM}}(W)$, then there exists a unique $\chi \in \mathcal{F}$ with minimal b -invariant.*
- (b) *If $\gamma \in \text{Cons}_\varphi^{\text{CM}}(W)$, then there exists a unique irreducible constituent χ of γ with minimal b -invariant.*

The next theorem is proved in [Lu2, Theorem 5.25 and its proof] (see also [Lu1] for the first occurrence of the *special* representations):

Theorem (Lusztig). *Assume that φ is constant. Then:*

- (a) *If $\mathcal{F} \in \text{Fam}_\varphi^{\text{Lus}}(W)$, then there exists a unique $\chi_{\mathcal{F}} \in \mathcal{F}$ with minimal b -invariant ($\chi_{\mathcal{F}}$ is called the **special** character of \mathcal{F}).*
- (b) *If $\gamma \in \text{Cons}_\varphi^{\text{Lus}}(\mathcal{F})$, then $\chi_{\mathcal{F}}$ is an irreducible constituent of γ (and, of course, among the irreducible constituents of γ , $\chi_{\mathcal{F}}$ is the unique one with minimal b -invariant).*

It turns out that, for general φ , there might exist Lusztig φ -families \mathcal{F} such that no element of \mathcal{F} occurs as an irreducible constituent of *all* the φ -constructible characters in $\text{Cons}_\varphi^{\text{Lus}}(\mathcal{F})$ (this already occurs in type B_3 , and the reader can also check this fact in type F_4 , using the tables given by Geck [Ge, Table 2]). Nevertheless, we will prove in this paper the following result, which is compatible with the above conjecture and the above theorems:

Theorem L. *Let $\varphi : S \rightarrow \mathbb{R}_{>0}$ be a weight function. Then:*

- (a) *If $\mathcal{F} \in \text{Fam}_{\varphi}^{\text{Lus}}(W)$, then there exists a unique $\chi \in \mathcal{F}$ with minimal b -invariant.*
- (b) *If $\gamma \in \text{Cons}_{\varphi}^{\text{Lus}}(W)$, then there exists a unique irreducible constituent χ of γ with minimal b -invariant.*

The proof of Theorem CM is general and conceptual, while our proof of Theorem L goes through a case-by-case analysis, based on Lusztig's description of φ -constructible characters and Lusztig φ -families [Lu3, §22].

REMARK 0 - Let γ_{χ} denote the coefficient of $\mathbf{t}^{b_{\chi}}$ in $F_{\chi}(\mathbf{t})$. Then it has been noticed by Lusztig [Lu1, §2, Page 325] that $\gamma_{\chi} = 1$ whenever χ is special.

As the only irreducible Coxeter systems affording possibly unequal parameters are of type $I_2(2m)$, F_4 or B_n , and as $\gamma_{\chi} = 1$ for any character χ in these groups, we can conclude that, in general (equal or unequal parameters), $\gamma_{\chi} = 1$ for all the characters χ with minimal b -invariant constructed in Theorem L (for both (a) and (b)).

The same property holds for the characters χ with minimal b -invariant constructed in Theorem CM (in this case, the proof is again general and conceptual [BoRo2]). ■

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1. Proof of Theorem L

1.A. Reduction. — It is easily seen that the proof of Theorem L may be reduced to the case where (W, S) is irreducible. If W is of type A_n , D_n , E_6 , E_7 , E_8 , H_3 or H_4 , then φ is necessarily constant and Theorem L follows immediately from Lusztig's Theorem. If W is dihedral, then Theorem L is easily checked using [Lu3, §17 and Lemma 22.2]. If W is of type F_4 , then Theorem L follows from inspection of [Ge, Table 2]. Therefore, this shows that we may, and we will, assume that W is of type B_n , with $n \geq 2$. Write $S = \{t, s_1, s_2, \dots, s_{n-1}\}$ in such a way that the Dynkin diagram of (W, S) is

$$(\#) \quad \begin{array}{ccccccc} & t & & s_1 & & s_2 & & \dots & & s_{n-1} \\ & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \dots & \text{---} & \circ \end{array}$$

Write $b = \varphi(t)$ and $a = \varphi(s_1) = \varphi(s_2) = \dots = \varphi(s_{n-1})$. If $b \notin a\mathbb{N}^*$, then $\text{Cons}_{\varphi}^{\text{Lus}}(W) = \text{Irr}(W)$ (see [Lu3, Proposition 22.25]) and Theorem L becomes obvious. So we may assume that $b = ra$ with $r \in \mathbb{N}^*$. Therefore:

Hypothesis. From now on, and until the end of this section, we will assume that (W, S) is of type B_n , with $n \geq 2$, that $S = \{t, s_1, s_2, \dots, s_{n-1}\}$ is such that the Dynkin diagram of (W, S) is given by $(\#)$, that $\varphi(t) = r\varphi(s_1) = r\varphi(s_2) = \dots = r\varphi(s_{n-1}) = 1$ with $r \in \mathbb{N}^*$.

1.B. Admissible involutions. — Let $l \geq 0$ and let Z be a totally ordered set of size $2l + r$. We shall define by induction on l what is an r -admissible involution of Z . Let $\iota : Z \rightarrow Z$ be an involution. Then ι is said r -admissible if it has r fixed points and, if $l \geq 1$, there exist two consecutive elements b and c of Z such that $\iota(b) = c$ and the restriction of ι to $Z \setminus \{b, c\}$ is r -admissible.

Note that, if ι is an r -admissible involution and if $\iota(b) = c > b$ and $\iota(z) = z$, then $z < b$ or $z > c$ (this is easily proved by induction on $|Z|$).

1.C. Symbols. — We shall denote by $\mathbf{Sym}_k(r)$ the set of symbols $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$ where $\beta = (\beta_1 < \beta_2 < \dots < \beta_{k+r})$ and $\gamma = (\gamma_1 < \gamma_2 < \dots < \gamma_k)$ are increasing sequences of non-zero natural numbers. We set

$$|\Lambda| = \sum_{i=1}^{k+r} (\beta_i - i) + \sum_{j=1}^k (\gamma_j - j)$$

and
$$\mathbf{b}(\Lambda) = \sum_{i=1}^{k+r} (2k + 2r - 2i)(\beta_i - i) + \sum_{j=1}^k (2k + 1 - 2j)(\gamma_j - j).$$

The number $\mathbf{b}(\Lambda)$ will be called the **b-invariant** of Λ . For simplifying our arguments, we shall define

$$\nabla_{k,r} = \sum_{i=1}^{k+r} (2k + 2r - 2i)i + \sum_{j=1}^k (2k + 1 - 2j)j$$

so that

$$\mathbf{b}(\Lambda) = \sum_{i=1}^{k+r} (2k + 2r - 2i)\beta_i + \sum_{j=1}^k (2k + 1 - 2j)\gamma_j - \nabla_{k,r}.$$

By abuse of notation, we denote by $\beta \cap \gamma$ the set $\{\beta_1, \beta_2, \dots, \beta_{k+r}\} \cap \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ and by $\beta \cup \gamma$ the set $\{\beta_1, \beta_2, \dots, \beta_{k+r}\} \cup \{\gamma_1, \gamma_2, \dots, \gamma_k\}$. We also set $\beta \dot{+} \gamma = (\beta \cup \gamma) \setminus (\beta \cap \gamma)$.

Now, let $\mathbf{z}'(\Lambda) = (\beta_1, \beta_2, \dots, \beta_r, \gamma_1, \beta_{r+1}, \gamma_2, \beta_{r+2}, \dots, \gamma_k, \beta_{r+k})$. We shall write

$$\mathbf{z}'(\Lambda) = (z'_1(\Lambda), z'_2(\Lambda), \dots, z'_{2k+r}(\Lambda)),$$

so that

$$\begin{aligned}
 \mathbf{b}(\Lambda) &= \sum_{i=1}^r (2k+2r-2i)z'_i(\Lambda) + \sum_{i=r+1}^{2k+r} (2k+r-i)z'_i(\Lambda) - \nabla_{k,r} \\
 (\clubsuit) \quad &= \sum_{i=1}^r (r-i)z'_i(\Lambda) + \sum_{i=1}^{2k+r} (2k+r-i)z'_i(\Lambda) - \nabla_{k,r} \\
 &= \sum_{i=1}^{r-1} \left(\sum_{j=1}^i z'_j(\Lambda) \right) + \sum_{i=1}^{2k+r-1} \left(\sum_{j=1}^i z'_j(\Lambda) \right) - \nabla_{k,r}.
 \end{aligned}$$

1.D. Families of symbols. — We denote by $\mathbf{z}(\Lambda)$ the sequence $z_1 \leq z_2 \leq \dots \leq z_{2k+r}$ obtained after rewriting the sequence $(\beta_1, \beta_2, \dots, \beta_{k+r}, \gamma_1, \gamma_2, \dots, \gamma_k)$ in non-decreasing order.

REMARK 1 - Note that the sequence $\mathbf{z}'(\Lambda)$ determines the symbol Λ , contrarily to the sequence $\mathbf{z}(\Lambda)$. However, $\mathbf{z}(\Lambda)$ determines completely $|\Lambda|$ thanks to the formula $|\Lambda| = \sum_{z \in \mathbf{z}(\Lambda)} z - r(r+1)/2 - (k+r)(k+r+1)/2$. \square

We say that two symbols $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$ and $\Lambda' = \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix}$ in $\mathbf{Sym}_k(r)$ are in the same family if $\mathbf{z}(\Lambda) = \mathbf{z}(\Lambda')$. Note that this is equivalent to say that $\beta \cap \gamma = \beta' \cap \gamma'$ and $\beta \cup \gamma = \beta' \cup \gamma'$. If \mathcal{F} is the family of Λ , we set $X_{\mathcal{F}} = \beta \cap \gamma$ and $Z_{\mathcal{F}} = \beta \dot{+} \gamma$: note that $X_{\mathcal{F}}$ and $Z_{\mathcal{F}}$ depend only on \mathcal{F} (and not on the particular choice of $\Lambda \in \mathcal{F}$).

If ι is an r -admissible involution of $Z_{\mathcal{F}}$, we denote by \mathcal{F}_{ι} the set of symbols $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$ in \mathcal{F} such that $|\beta \cap \omega| = 1$ for all ι -orbits ω .

1.E. Lusztig families, constructible characters. — Let $\Lambda \in \mathbf{Sym}_k(r)$ be such that $|\Lambda| = n$. Let $\text{Bip}(n)$ be the set of bipartitions of n . We set

$$\lambda_1(\Lambda) = (\beta_{k+r} - (k+r) \geq \dots \geq \beta_2 - 2 \geq \beta_1 - 1),$$

$$\lambda_2(\Lambda) = (\gamma_k - k \geq \dots \geq \gamma_2 - 2 \geq \gamma_1 - 1)$$

and

$$\lambda(\Lambda) = (\lambda_1(\Lambda), \lambda_2(\Lambda)).$$

Then $\lambda(\Lambda)$ is a bipartition of n . We denote by χ_{Λ} the irreducible character of W denoted by $\chi_{\lambda(\Lambda)}$ in [Lu3, §22] or in [GePf, §5.5.3]. Then [GePf, §5.5.3]

$$(\diamond) \quad b_{\chi_{\Lambda}} = \mathbf{b}(\Lambda).$$

With these notation, Lusztig described the φ -constructible characters in [Lu3, Proposition 22.24], from which the description of Lusztig φ -families follow by using [Lu3, Lemma 22.22]:

Theorem 2 (Lusztig). Let \mathcal{F}_{Lus} be a Lusztig φ -family and let $\gamma \in \text{Cons}_{\varphi}^{\text{Lus}}(\mathcal{F}_{\text{Lus}})$. If we choose k sufficiently large, then:

(a) There exists a family \mathcal{F} of symbols in $\mathbf{Sym}_k(r)$ such that

$$\mathcal{F}_{\text{Lus}} = \{\chi_{\Lambda} \mid \Lambda \in \mathcal{F}\}.$$

(b) There exists an r -admissible involution ι of $Z_{\mathcal{F}}$ such that

$$\gamma = \sum_{\Lambda \in \mathcal{F}_{\iota}} \chi_{\Lambda}.$$

If $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$, we set $\Lambda^{\#} = \begin{pmatrix} \beta \setminus (\beta \cap \gamma) \\ \gamma \setminus (\beta \cap \gamma) \end{pmatrix}$.

Definition 3. The symbol Λ is said *special* if $\mathbf{z}(\Lambda^{\#}) = \mathbf{z}'(\Lambda^{\#})$.

REMARK 4 - According to Remark 1, there is a unique special symbol in each family. It will be denoted by $\Lambda_{\mathcal{F}}$. Finally, note that, if Λ, Λ' belong to the same family, then $|\Lambda| = |\Lambda'|$. \square

Now, Theorem L follows from Theorem 2, Formula (\diamond) and the following next Theorem:

Theorem 5. Let \mathcal{F} be a family of symbols in $\mathbf{Sym}_k(r)$, let ι be an r -admissible involution of $Z_{\mathcal{F}}$ and let $\Lambda \in \mathcal{F}$. Then:

- (a) $\mathbf{b}(\Lambda) \geq \mathbf{b}(\Lambda_{\mathcal{F}})$ with equality if and only if $\Lambda = \Lambda_{\mathcal{F}}$.
- (b) There is a unique symbol $\Lambda_{\mathcal{F}, \iota}$ in \mathcal{F}_{ι} such that, if $\Lambda \in \mathcal{F}_{\iota}$, then $\mathbf{b}(\Lambda) \geq \mathbf{b}(\Lambda_{\mathcal{F}, \iota})$, with equality if and only if $\Lambda = \Lambda_{\mathcal{F}, \iota}$.

The rest of this section is devoted to the proof of Theorem 5.

1.F. Reduction for the proof of Theorem 5. — First, assume that $X_{\mathcal{F}} \neq \emptyset$. Let $b \in X_{\mathcal{F}}$ and let $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \in \mathcal{F}$. Then $b \in \beta \cap \gamma = X_{\mathcal{F}}$ and we denote by $\beta[b]$ the sequence obtained by removing b to β . Similarly, let $\Lambda[b] = \begin{pmatrix} \beta[b] \\ \gamma[b] \end{pmatrix}$.

Then $\Lambda[b] \in \mathbf{Sym}_{k-1}(r)$ and

$$(\heartsuit) \quad \mathbf{b}(\Lambda) = \mathbf{b}(\Lambda[b]) + \nabla_{k,r} - \nabla_{k-1,r} + b \left(4k + 2r + 1 - \sum_{\substack{z \in \mathbf{z}(\Lambda) \\ z \leq b}} 2 \right) + 2 \sum_{\substack{z \in \mathbf{z}(\Lambda) \\ z < b}} z.$$

Proof of (♥). Let i_0 and j_0 be such that $\beta_{i_0} = b$ and $\gamma_{j_0} = b$. Then

$$\mathbf{b}(\Lambda) - \mathbf{b}(\Lambda[b]) = \nabla_{k,r} - \nabla_{k-1,r} + (2k + 2r - 2i_0)b + \sum_{i=1}^{i_0-1} 2\beta_i + (2k + 1 - 2j_0)b + \sum_{j=1}^{j_0-1} 2\gamma_j.$$

But the numbers $\beta_1, \beta_2, \dots, \beta_{i_0}, \gamma_1, \gamma_2, \dots, \gamma_{j_0}$ are exactly the elements of the sequence $\mathbf{z}(\Lambda)$ which are $\leq b$. So

$$i_0 + j_0 = \sum_{\substack{z \in \mathbf{z}(\Lambda) \\ z \leq b}} 1$$

and

$$\sum_{i=1}^{i_0-1} \beta_i + \sum_{j=1}^{j_0-1} \gamma_j = \sum_{\substack{z \in \mathbf{z}(\Lambda) \\ z < b}} z.$$

This shows (♥). ■

Now, the family of $\Lambda[b]$ depends only on the family of Λ (and not on Λ itself): indeed, $\mathbf{z}(\Lambda[b])$ is obtained from $\mathbf{z}(\Lambda)$ by removing the two entries equal to b . We will denote by $\mathcal{F}[b]$ the family of $\Lambda[b]$. Moreover, $Z_{\mathcal{F}[b]} = Z_{\mathcal{F}}$ and the map $\Lambda \mapsto \Lambda[b]$ induces a bijection between \mathcal{F} and $\mathcal{F}[b]$, and also induces a bijection between \mathcal{F}_i and $\mathcal{F}[b]_i$.

On the other hand, the formula (♥) shows that the difference between $\mathbf{b}(\Lambda)$ and $\mathbf{b}(\Lambda[b])$ depends only on b and \mathcal{F} , so proving Theorem 5 for the pair (\mathcal{F}, ι) is equivalent to proving Theorem 5 for the pair $(\mathcal{F}[b], \iota)$. By applying several times this principle if necessary, this means that we may, and we will, assume that

$$X_{\mathcal{F}} = \emptyset.$$

1.G. Proof of Theorem 5(a). — First, note that $\mathbf{z}(\Lambda) = \mathbf{z}(\Lambda_{\mathcal{F}}) = \mathbf{z}'(\Lambda_{\mathcal{F}})$ (the last equality follows from the fact that $\Lambda_{\mathcal{F}}$ is special and $X_{\mathcal{F}} = \emptyset$). As $\mathbf{z}'(\Lambda)$ is a permutation of the non-decreasing sequence $\mathbf{z}'(\Lambda_{\mathcal{F}})$, we have

$$\sum_{j=1}^i z'_j(\Lambda) \geq \sum_{j=1}^i z'_j(\Lambda_{\mathcal{F}})$$

for all $i \in \{1, 2, \dots, 2k + r\}$. So, it follows from (♣) that

$$\mathbf{b}(\Lambda) - \mathbf{b}(\Lambda_{\mathcal{F}}) = \sum_{i=1}^{r-1} \left(\sum_{j=1}^i (z'_j(\Lambda) - z'_j(\Lambda_{\mathcal{F}})) \right) + \sum_{i=1}^{2k+r-1} \left(\sum_{j=1}^i (z'_j(\Lambda) - z'_j(\Lambda_{\mathcal{F}})) \right).$$

So $\mathbf{b}(\Lambda) \geq \mathbf{b}(\Lambda_{\mathcal{F}})$ with equality only whenever $\sum_{j=1}^i z'_j(\Lambda) = \sum_{j=1}^i z'_j(\Lambda_{\mathcal{F}})$ for all $i \in \{1, 2, \dots, 2k + r\}$. The proof of Theorem 5(a) is complete.

1.H. Proof of Theorem 5(b). — We denote by $f_r < \dots < f_1$ the elements of $Z_{\mathcal{F}}$ which are fixed by ι . We also set $f_{r+1} = 0$ and $f_0 = \infty$. As ι is r -admissible, the set $Z_{\mathcal{F}}^{(d)} = \{z \in Z_{\mathcal{F}} \mid f_{d+1} < z < f_d\}$ is ι -stable and contains no ι -fixed point (for $d \in \{0, 1, \dots, r\}$). Let $k_d = |Z_{\mathcal{F}}^{(d)}|/2$ and let ι_d be the restriction of ι to $Z_{\mathcal{F}}^{(d)}$. Then ι_d is a 0-admissible involution of $Z_{\mathcal{F}}^{(d)}$.

If $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \in \mathcal{F}_{\iota}$, we set $\beta^{(d)} = \beta \cap Z_{\mathcal{F}}^{(d)}$, $\gamma^{(d)} = \gamma \cap Z_{\mathcal{F}}^{(d)}$ and $\Lambda^{(d)} = \begin{pmatrix} \beta^{(d)} \\ \gamma^{(d)} \end{pmatrix}$. Then $\Lambda^{(d)} \in \mathbf{Sym}_{k_d}(0)$ and, if $\mathcal{F}^{(d)}$ denotes the family of $\Lambda^{(d)}$, then $\Lambda^{(d)} \in \mathcal{F}_{\iota_d}^{(d)}$.

Now, if $\Lambda' = \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix} \in \mathbf{Sym}_{k'}(0)$, we set

$$\mathbf{b}_d(\Lambda') = \sum_{i=1}^{k'} (2k' + 2d - 2i) \beta'_i + \sum_{j=1}^{k'} (2k' + 1 - 2j) \gamma'_j.$$

The number $\mathbf{b}_d(\Lambda')$ is called the \mathbf{b}_d -invariant of Λ' . It then follows from the definition of \mathbf{b} and $\nabla_{k,r}$ that

$$(\spadesuit) \quad \mathbf{b}(\Lambda) = \sum_{d=0}^r \mathbf{b}_d(\Lambda^{(d)}) - \nabla_{k,r} + \sum_{d=1}^r 2(k_0 + k_1 + \dots + k_{d-1}) \left(f_d + \sum_{z \in Z^{(d)}} z \right).$$

Since the map

$$\begin{aligned} \mathcal{F}_{\iota} &\longrightarrow \prod_{d=0}^r \mathcal{F}_{\iota_d}^{(d)} \\ \Lambda &\longmapsto (\Lambda^{(0)}, \Lambda^{(1)}, \dots, \Lambda^{(d)}) \end{aligned}$$

is bijective and since $\mathbf{b}(\Lambda) - \sum_{d=0}^r \mathbf{b}_d(\Lambda^{(d)})$ depends only on (\mathcal{F}, ι) and not on Λ (as shown by the formula (\spadesuit)), Theorem 5(b) will follow from the following lemma :

Lemma 6. *There exists a unique symbol in $\mathcal{F}_{\iota_d}^{(d)}$ with minimal \mathbf{b}_d -invariant.*

The proof of Lemma 6 will be given in the next section.

2. Minimal \mathbf{b}_d -invariant

For simplifying notation, we set $Z = Z_{\mathcal{F}}^{(d)}$, $l = k_d$, $\mathcal{G} = \mathcal{F}^{(d)}$ and $J = \iota_d$. Let us write $Z = \{z_1, z_2, \dots, z_{2l}\}$ with $z_1 < z_2 < \dots < z_{2l}$. Recall from the previous section that J is a 0-admissible involution of Z .

2.A. Construction. — We shall define by induction on $l \geq 0$ a symbol $\Lambda_j^{(d)}(Z) \in \mathcal{G}_j$. If $l = 0$, then $\Lambda_j^{(d)}(Z)$ is obviously empty. So assume now that, for any set of non-zero integers Z' of order $2(l - 1)$, for any 0-admissible involution j' of Z' and any $d' \geq 0$, we have defined a symbol $\Lambda_{j'}^{(d')}(Z')$. Then $\Lambda_j^{(d)}(Z) = \begin{pmatrix} \beta_j^{(d)}(Z) \\ \gamma_j^{(d)}(Z) \end{pmatrix}$ is defined as follows: let $Z' = Z \setminus \{z_1, \iota(z_1)\}$, j' the restriction of j to Z' and let

$$d' = \begin{cases} d - 1 & \text{if } d \geq 1, \\ 1 & \text{if } d = 0. \end{cases}$$

Then $|Z'| = 2(l - 1)$ and j' is 0-admissible. So $\Lambda_{j'}^{(d')}(Z') = \begin{pmatrix} \beta_{j'}^{(d')}(Z') \\ \gamma_{j'}^{(d')}(Z') \end{pmatrix}$ is well-defined by the induction hypothesis. We then set

$$\beta_j^{(d)}(Z) = \begin{cases} \beta_{j'}^{(d')}(Z') \cup \{z_1\} & \text{if } d \geq 1, \\ \beta_{j'}^{(d')}(Z') \cup \{J(z_1)\} & \text{if } d = 0, \end{cases}$$

and

$$\gamma_j^{(d)}(Z) = \begin{cases} \gamma_{j'}^{(d')}(Z') \cup \{J(z_1)\} & \text{if } d \geq 1, \\ \gamma_{j'}^{(d')}(Z') \cup \{z_1\} & \text{if } d = 0. \end{cases}$$

Then Lemma 6 is implied by the next lemma :

Lemma 6⁺. *Let $\Lambda \in \mathcal{G}_j$. Then $\mathbf{b}_d(\Lambda) \geq \mathbf{b}_d(\Lambda_j^{(d)}(Z))$ with equality if and only if $\Lambda = \Lambda_j^{(d)}(Z)$.*

The rest of this section is devoted to the proof of Lemma 6⁺. We will first prove Lemma 6⁺ whenever $d \in \{0, 1\}$ using Lusztig's Theorem. We will then turn to the general case, which will be handled by induction on $l = |Z|/2$. We fix $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \in \mathcal{G}_l$.

2.B. Proof of Lemma 6⁺ whenever $d = 1$. — Let z be a natural number strictly bigger than all the elements of Z . Let $\tilde{\Lambda} = \begin{pmatrix} \beta \cup \{z\} \\ \gamma \end{pmatrix} \in \mathbf{Sym}_k(1)$. Then $\mathbf{b}_1(\Lambda) = \mathbf{b}(\tilde{\Lambda}) + C$, where C depends only on Z . Let $\tilde{\Lambda}_0 = \begin{pmatrix} z_1, z_3, \dots, z_{2l-1}, z \\ z_2, \dots, z_{2l} \end{pmatrix}$. Since j is 0-admissible, it is easily seen that, if $j(z_i) = z_j$, then $j - i$ is odd. So $\tilde{\Lambda}_0 \in \mathcal{G}_j$. But, by [Lu1, §5], $\mathbf{b}(\tilde{\Lambda}) \geq \mathbf{b}(\tilde{\Lambda}_0)$ with equality if and only if $\tilde{\Lambda} = \tilde{\Lambda}_0$. So it is sufficient to notice that $\Lambda_j^{(1)}(Z) = \tilde{\Lambda}_0$, which is easily checked.

2.C. Proof of Lemma 6⁺ whenever $d = 0$. — Assume in this subsection, and only in this subsection, that $d = 0$ or 1. We denote by $\Lambda^{\text{op}} = \begin{pmatrix} \gamma \\ \beta \end{pmatrix} \in \mathcal{G}_j$. It is readily seen from the construction that $\Lambda_j^{(0)}(Z)^{\text{op}} = \Lambda_j^{(1)}(Z)$ and that

$$\mathbf{b}_1(\Lambda) = \mathbf{b}_0(\Lambda^{\text{op}}) + \sum_{z \in Z} z.$$

So Lemma 6⁺ for $d = 0$ follows from Lemma 6⁺ for $d = 1$.

2.D. Proof of Lemma 6⁺ whenever $d \geq 2$. — Assume now, and until the end of this section, that $d \geq 2$. We shall prove Lemma 6⁺ by induction on $l = |Z|/2$. The result is obvious if $l = 0$, as well as if $l = 1$. So we assume that $l \geq 2$ and that Lemma 6⁺ holds for $l' \leq l-1$. Write $j(z_1) = z_{2m}$, where $m \leq l$ (note that $j(z_1) \notin \{z_1, z_3, z_5, \dots, z_{2l-1}\}$ since j is 0-admissible).

Assume first that $m < l$. Then Z can be written as the union $Z = Z^+ \dot{\cup} Z^-$, where $Z^+ = \{z_1, z_2, \dots, z_{2m}\}$ and $Z^- = \{z_{2m+1}, z_{2m+2}, \dots, z_{2l}\}$ are j -stable (since j is 0-admissible). If $\varepsilon \in \{+, -\}$, let j^ε denote the restriction of j to Z^ε , let $\beta^\varepsilon = \beta \cap Z^\varepsilon$, $\gamma^\varepsilon = \gamma \cap Z^\varepsilon$ and $\Lambda^\varepsilon = \begin{pmatrix} \beta^\varepsilon \\ \gamma^\varepsilon \end{pmatrix}$, and let \mathcal{G}^ε denote the family of Λ^ε . Then it is easily seen that $\Lambda^\varepsilon \in \mathcal{G}_{j^\varepsilon}^\varepsilon$, that $\mathbf{b}_d(\Lambda) - (\mathbf{b}_d(\Lambda^+) + \mathbf{b}_d(\Lambda^-))$ depends only on (\mathcal{G}, j) and that $\Lambda_j^{(d)}(Z)^\varepsilon = \Lambda_{j^\varepsilon}^{(d)}(Z^\varepsilon)$. By the induction hypothesis, $\mathbf{b}_d(\Lambda^\varepsilon) \geq \mathbf{b}_d(\Lambda_{j^\varepsilon}^{(d)}(Z^\varepsilon))$ with equality if and only if $\Lambda^\varepsilon = \Lambda_{j^\varepsilon}^{(d)}(Z^\varepsilon)$. So the result follows in this case. This means that we may, and we will, work under the following hypothesis:

Hypothesis. From now on, and until the end of this section, we assume that $j(z_1) = z_{2l}$.

As in the construction of $\Lambda_j^{(d)}(Z)$, let $Z' = Z \setminus \{z_1, z_{2l}\} = \{z_2, z_3, \dots, z_{2l-1}\}$, let j' denote the restriction of j to Z' and let

$$d' = \begin{cases} d-1 & \text{if } d \geq 1, \\ 1 & \text{if } d = 0. \end{cases}$$

Then $|Z'| = 2(l-1)$ and j' is 0-admissible. Let $\Lambda' = \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix}$ where $\beta' = \beta \setminus \{z_1, z_{2l}\}$ and $\gamma' = \gamma \setminus \{z_1, z_{2l}\}$. Since $d \geq 2$, we have $z_1 \in \beta_j^{(d)}(Z)$ and $z_{2l} \in \gamma_j^{(d)}(Z)$. This implies that

$$(\star) \quad \mathbf{b}_d(\Lambda_j^{(d)}(Z)) = \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) + z_{2l} + 2(l+d)z_1 + 2 \sum_{z \in Z'} z.$$

If $z_1 \in \beta$, then $\Lambda = \Lambda_j^{(d)}(Z)$ if and only if $\Lambda' = \Lambda_{j'}^{(d')}(Z')$ and again

$$\mathbf{b}_d(\Lambda) = \mathbf{b}_{d-1}(\Lambda') + z_{2l} + 2(l+d)z_1 + 2 \sum_{z \in Z'} z.$$

So the result follows from (★) and from the induction hypothesis.

This means that we may, and we will, assume that $z_1 \in \gamma$. In this case,

$$\mathbf{b}_d(\Lambda) = \mathbf{b}_{d+1}(\Lambda') + 2dz_{2l} + (2l+1)z_1.$$

Then it follows from (★) that

$$\mathbf{b}_d(\Lambda) - \mathbf{b}_d(\Lambda_j^{(d)}(Z)) = \mathbf{b}_{d+1}(\Lambda') - \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) + (2d-1)(z_{2l} - z_1) - 2 \sum_{z \in Z'} z.$$

So, by the induction hypothesis,

$$\mathbf{b}_d(\Lambda) - \mathbf{b}_d(\Lambda_j^{(d)}(Z)) \geq \mathbf{b}_{d+1}(\Lambda_{j'}^{(d+1)}(Z')) - \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) + (2d-1)(z_{2l} - z_1) - 2 \sum_{z \in Z'} z.$$

Since $z_{2l} - z_1 > z_{2l-1} - z_2$, it is sufficient to show that

$$(?) \quad \mathbf{b}_{d+1}(\Lambda_{j'}^{(d+1)}(Z')) - \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) \geq -(2d-1)(z_{2l-1} - z_2) + 2 \sum_{z \in Z'} z.$$

This will be proved by induction on the size of Z' . First, if $J(z_2) < z_{2l}$, then we can separate Z' into two j' -stable subsets and a similar argument as before allows to conclude thanks to the induction hypothesis.

So we assume that $J'(z_2) = z_{2l-1}$. Let $Z'' = Z' \setminus \{z_2, z_{2l-1}\}$ and let j'' denote the restriction of j' to Z'' . Since $z_2 \in \beta_{j'}^{(d+1)}(Z')$, we can apply (★) one step further to get

$$\begin{aligned} \mathbf{b}_{d+1}(\Lambda_{j'}^{(d+1)}(Z')) - \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) &= \mathbf{b}_d(\Lambda_{j''}^{(d)}(Z'') + z_{2l-1} + 2(l+d)z_2 + 2 \sum_{z \in Z''} z \\ &\quad - (\mathbf{b}_{d-2}(\Lambda_{j''}^{(d-2)}(Z'')) + z_{2l-1} + 2(l+d-2)z_2 + 2 \sum_{z \in Z''} z) \\ &= \mathbf{b}_d(\Lambda_{j''}^{(d)}(Z'')) - \mathbf{b}_{d-2}(\Lambda_{j''}^{(d-2)}(Z'')) + 4z_2. \end{aligned}$$

So, by the induction hypothesis,

$$\begin{aligned} \mathbf{b}_{d+1}(\Lambda_{j'}^{(d+1)}(Z')) - \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) &\geq -(2d-3)(z_{2l-2} - z_3) + 2 \sum_{z \in Z''} z + 4z_2 \\ &\geq -(2d-3)(z_{2l-1} - z_2) + 2 \sum_{z \in Z'} z + 2z_2 - 2z_{2l-1} \\ &= -(2d-1)(z_{2l-1} - z_2) + 2 \sum_{z \in Z'} z, \end{aligned}$$

as desired. This shows (?) and completes the proof of Lemma 6⁺.

3. Complex reflection groups

If \mathcal{W} is a complex reflection group, then R. Rouquier and the author have also defined Calogero-Moser cellular characters and Calogero-Moser families (see [BoRo1] or [BoRo2]). If \mathcal{W} is of type $G(l, 1, n)$ (in Shephard-Todd classification), then Leclerc and Miyachi [LeMi, §6.3] proposed, in link with canonical bases of $U_v(\mathfrak{sl}_\infty)$ -modules, a family of characters that could be good analogue of constructible characters: let us call them the *Leclerc-Miyachi constructible characters* of $G(l, 1, n)$. If $l = 2$, then they coincide with constructible characters [LeMi, Theorem 10].

Of course, it would be interesting to know if Calogero-Moser cellular characters coincide with the Leclerc-Miyachi ones: this seems rather complicated but it should be at least possible to check if the Leclerc-Miyachi constructible characters satisfy the analogous properties with respect to the b -invariant.

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